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Definitions and Purpose.

A *composite line* may be defined as an electrically conducting line formed of two or more successive sections, each section having its own length and its own particular uniformly distributed resistance, inductance, capacitance, and leakance. Each such section, considered separately, may be described as a *single line*. A composite line is, therefore, a successive connection of single lines which differ in linear constants.

It has been shown by the writer in a preceding paper[1] that any uniform single line, operated in the steady state, either by single-frequency alternating currents or by continuous currents, may be externally imitated by a symmetrical triple conductor. The triple conductor which can be substituted for a single line in a steady system of electric flow without disturbing the potentials, or currents, at or outside of the line terminals, may be defined as an *equivalent circuit* of the line. A star-connected equivalent circuit, with two equal line branches and a single leak, may be called an equivalent T; while a delta-connected equivalent circuit with two equal leaks, and a single line-resistance or impedance between them, may be called an equivalent Π. It is the object of this paper to extend the laws of equivalent circuits from single lines to composite lines, with or without loads, and also to present formulas for the distribution of current and potential over such composite lines.

Important Practical Application of the Problem.

An important application of this problem is found in telephony. With given sending and receiving apparatus, the commercial operativeness of a telephonic metallic circuit apparently depends only on the strength of alternating current, at a certain standard frequency, in

[1] "Artificial Lines for Continuous Currents in the Steady State." See appended Bibliography.

the receiver. That is, it depends on the "receiving-end impedance" of the circuit, or the ratio of the impressed standard-frequency alternating emf. at the sending end, to the current-strength at the receiving end. If this receiving-end impedance of the circuit, including the impedance of the receiving apparatus, is not greater than 25,000 ohms (12,500 ohms per wire), at the angular velocity $\omega = 5000$ radians per second, commercial telephony will readily be possible with the standard Bell telephone apparatus used in the United States; unless the distortion of the speech-waves, due to unequal attenuation at different frequencies, is unusually great. If the circuit receiving-end impedance exceeds 200,000 ohms (100,000 ohms per wire) at $\omega = 5000$ radians per second, even expert telephonists will ordinarily be unable to converse with this apparatus over the line.

It is easy, with the aid of formulas given in the above-mentioned preceding paper, to find the equivalent Π of a simple single telephone line of given length, uniform linear constants, and assigned terminal conditions. But for most practical purposes this is not enough. Most long telephone lines in practical service are not single, but composite. Consider the case of a subscriber A, in Boston, talking to a subscriber B, in New York. First there is the terminal apparatus at A; then, say, a few kilometers of underground line in Boston. Next comes the long-distance overhead line from Boston to New York, perhaps consisting of more than one section and size of wire. Then come one or more sections of underground wire in New York, before we end the circuit in B's apparatus. At two or three intermediate exchanges in this circuit there may also be casual loads, formed by supervisory relays, or other instruments. The critical receiving-end impedance must not be exceeded in this composite circuit, if the talking is to be of satisfactory quality. Actual trial of the line by conversation will determine, with a fair degree of precision, whether the limiting permissible receiving-end impedance has been exceeded by the line. But the designing telephone engineer seeks to know, in advance, whether a certain projected composite line will, when constructed, fall within the permissible limit of receiving-end impedance. If working formulas can be developed, that are not too lengthy and complicated, for determining the receiving-end impedance of composite lines, they may help the designing engineer to decide questions of line construction.

In this paper the discussion will be principally confined to direct-current composite lines. The formulas thus derived are all easily presented, grasped, and checked by Ohm's law, since they involve only real numerical quantities. In the direct-current case the hyperbolic quantities used are all functions of simple real numerics, for which

published tables are available. Identically the same formulas are, however, applicable to single-frequency alternating-current cases, by expanding their interpretation from real to complex numbers; or from one space-dimension into two, using impedances for resistances and plane-vectors for potentials and currents. Unfortunately, however, we have no tables of hyperbolic sines, cosines, and tangents available, as yet, for complex arguments except for the particular case of semi-imaginaries,[2] or plane-vectors of 45°; so that in working out the alternating-current cases, as, for example, in telephony, the engineer is delayed by having to assume the duties of a computer, and to work out his own hyperbolic sines, cosines, and tangents. However, even thus handicapped, it is claimed that the formulas here presented will not be too lengthy for the engineer to use in important cases. If hyperbolic tables of complex arguments were worked out and published, the formulas could, with their help, be applied almost as quickly and conveniently to alternating-current cases as they can be applied at present

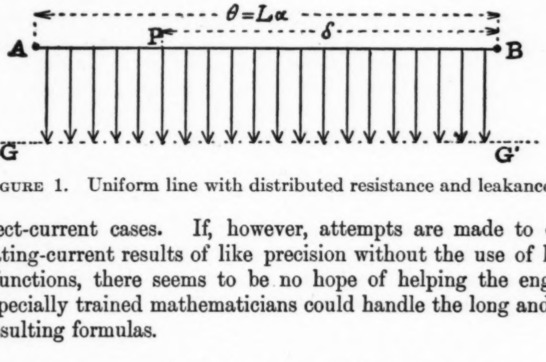

FIGURE 1. Uniform line with distributed resistance and leakance.

to direct-current cases. If, however, attempts are made to obtain alternating-current results of like precision without the use of hyperbolic functions, there seems to be no hope of helping the engineer. Only specially trained mathematicians could handle the long and complex resulting formulas.

PRELIMINARY REVIEW OF SINGLE-LINE FORMULAS.

In order to pass to composite lines, we may first briefly review the laws of equivalent circuits for single lines. The fundamental formulas will be given for direct-current (D. C.) and for alternating-current (A. C.) circuits, in parallel columns.

Let AB, Figure 1, be a uniform single line operated to ground, or zero-potential, return circuit.

L = the length of the line in kilometers (or miles).

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[2] See Table appended to "The Alternating-Current Theory of Transmission-Speed over Submarine Cables," referred to in the Bibliography.

$r =$ the linear resistance of the line (ohms per wire km.).
$g =$ the linear leakance of the line (mhos per wire km.).
$l =$ the linear inductance of the line (henrys per wire km.).
$c =$ the linear capacitance of the line (farads per wire km.).
$n =$ the frequency of the impressed emf. at A (cycles per second).
$\omega = 2\pi n$, the angular velocity of the impressed emf. at A (radians per second).
$j = \sqrt{-1}$.

The attenuation constant of the line is

$$\text{D. C.} \quad a = \sqrt{rg} \quad \frac{\text{hyps}}{\text{km.}}; \quad \text{A. C.} \quad a = \sqrt{(r + jl\omega)(g + jc\omega)} \ \frac{\text{hyps}}{\text{km.}} \angle. \tag{1}$$

In the D. C. case a is a real numerical quantity which we may, for convenience of subsequent operation, define as a "linear hyperbolic angle," or "hyperbolic angle" per km. of length. Although it is a simple numeric per unit length of line, yet, since it forms the basis of argument in hyperbolic tables, we may call it a "hyperbolic angle" per unit length of line, and denote a hyperbolic unit angle as a "hyp." In the A. C. case a is a plane-vector "hyperbolic angle," or complex quantity, per unit length of line.

The hyperbolic angle subtended by the line AB is

$$\text{D. C.} \quad \theta = La \quad \text{hyps}; \qquad \text{A. C.} \quad \theta = La \quad \text{hyps} \angle. \tag{2}$$

θ is a real numeric for the D. C. case, and a plane-vector, or numeric at a definite angle in the reference plane, for the A. C. case. The surge-resistance, or surge-impedance, of the line is

$$\text{D. C.} \quad z = \sqrt{\frac{r}{g}} \quad \text{ohms}; \qquad \text{A. C.} \quad z = \sqrt{\frac{r + jl\omega}{g + jc\omega}} \quad \text{ohms} \angle. \tag{3}$$

The *surge-impedance* of an A. C. line is the impedance that the line offers at any point of its length to the propagation of waves of the frequency considered. It is a vector resistance, or impedance, often closely approximating numerically to $\sqrt{l/c}$. The *surge-admittance* of a line is the reciprocal of its surge-impedance.

In wave-propagation theory, and also in the steady-state theory here considered, θ and z, the hyperbolic angle and surge-impedance of a line, are its fundamental characteristics; while r, g, l, and c are its secondary or incidental characteristics.

Single Line Freed at Distant End.

If the line AB is freed at B, its resistance at A, measured to ground, is

$$R_{fA} = z \coth \theta \qquad \text{ohms. (4)}$$

In the D. C. case the hyperbolic angle θ is a simple real quantity, z is a simple numerical resistance, and $\coth \theta$ is the hyperbolic cotangent of θ, a real numeric, obtainable from tables of hyperbolic functions. Consequently, R_{fA} is a simple resistance in ohms. In the A. C. case, however, z is an impedance, or vector resistance, θ is also a vector quantity, and the hyperbolic cotangent of this vector is not ordinarily obtainable from any tables thus far published. It must be computed, say, with the aid of formula (142). The product of z and this cotangent is, therefore, a vector resistance, or impedance, R_{fA}. Similarly, all the remaining formulas of this paper may be regarded as applying either to D. C. or to A. C. cases; but the D. C. reasoning will be followed, for simplicity of numerical check.

At any point P (Figure 1) along the line, distant l' km. from B, its hyperbolic angular distance from B will be

$$\delta = l'a \qquad \text{hyps. (5)}$$

The potential at P is

$$u_P = u_B \cosh \delta \qquad \text{volts, (6)}$$

where u_B is the potential at the far free end B, defined by the condition

$$u_B = u_A / \cosh \theta \qquad \text{volts; (7)}$$

whence

$$u_P = u_A \frac{\cosh \delta}{\cosh \theta} \qquad \text{volts. (8)}$$

The curve of potential, or voltage to ground, plotted as ordinates along the line AB is, therefore, a curve of hyp. cosines, or a catenary. In the A. C. case the curve of vector lengths, or numerical values, of potential, plotted as ordinates along AB, is a sinusoid superposed upon a catenary.

The current-strength at the point P is

$$\dot{i}_P = \dot{i}_A \frac{\sinh \delta}{\sinh \theta} \qquad \text{amperes, (9)}$$

where i_A is the current entering the line at A. The curve of current-strength plotted as ordinates along AB is, therefore, in the D. C. case, a curve of hyp. sines, or curve of catenary-slope.

The resistance of the line, at and beyond the point P, measured to ground is

$$R_{fP} = z \coth \delta \qquad \text{ohms, (10)}$$

or

$$R_{fP} = R_{fA} \frac{\coth \delta}{\coth \theta} \qquad \text{ohms. (11)}$$

Single Line Grounded at Distant End.

If the line, instead of being freed at B (Figure 1), is grounded at B, its resistance at A is

$$R_{gA} = z \tanh \theta \qquad \text{ohms. (12)}$$

At any point P, angularly distant δ hyps from B, the line resistance beyond P, measured to ground, is

$$R_{gP} = z \tanh \delta \qquad \text{ohms, (13)}$$

or

$$R_{gP} = R_{gA} \frac{\tanh \delta}{\tanh \theta} \qquad \text{ohms. (14)}$$

The potential at P, in terms of the potential u_A at A, is

$$u_P = u_A \frac{\sinh \delta}{\sinh \theta} \qquad \text{volts. (15)}$$

The current-strength at P, in terms of the current-strength i_A entering the line at A, is

$$i_P = i_A \frac{\cosh \delta}{\cosh \theta} \qquad \text{amperes. (16)}$$

For example, consider a line, AB, Figure 1, of $L = 100$ km., with a linear resistance r of 20 ohms per wire-km., and a linear leakance g of 20×10^{-6} mho per wire km. (20 micromhos per km.), corresponding to a linear insulation-resistance of 50,000 km-ohms. The attenuation-constant of this line is $\alpha = 2 \times 10^{-2}$ hyp. per km. by (1), and the hyperbolic angle subtended by the line is $\theta = 2$ hyps. by (2). The surge-resistance of the line is $z = 1000$ ohms by (3). Then the resistance offered by the line at A, when freed at B, is, by (4),

$$R_{fA} = 1000 \coth 2 = 1000 \times 1.037315 = 1{,}037.315 \text{ ohms,}$$

and when grounded at B, by (12),

$$R_{gA} = 1000 \tanh 2 = 1000 \times 0.964026 = 964.026 \text{ ohms.}$$

EQUIVALENT CIRCUITS OF SINGLE LINE.

The equivalent T of this line is a star-connection of three resistances AO, GO, BO (Figure 2), two of which — the line-branches AO, OB, — are equal; while OG is a leakage-resistance to ground. This equivalent T, when correctly proportioned, has the property of being able to replace the uniformly leaky line AB, without disturbing in any manner the system of potentials and currents outside the terminals ABG. Let $g' = 1/R'$ be the conductance of the leak OG′; then

$$g' = y \sinh \theta \quad \text{mhos,} \quad (17)$$

where $y = 1/z$ mhos, the reciprocal of the surge-resistance. We may call y the *surge-conductance* (A. C. *surge-admittance*).

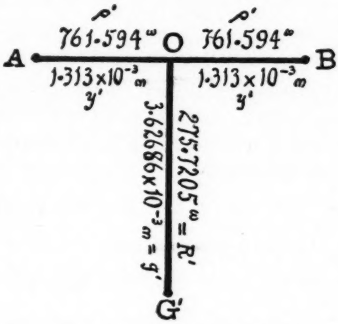

FIGURE 2. Equivalent T of uniform line.

Let ρ' be the resistance of each line-branch AO, OB; then

$$\rho' = z \tanh \frac{\theta}{2} = z \coth \theta - R' = R_f - R' \quad \text{mhos.} \quad (18)$$

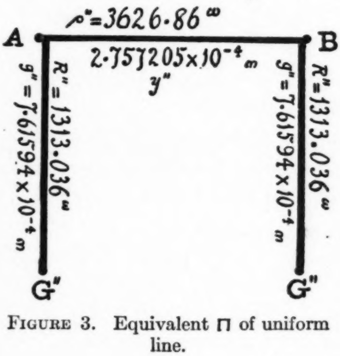

FIGURE 3. Equivalent Π of uniform line.

Thus, for the line above considered, $g' = 0.001 \times \sinh 2 = 0.001 \times 3.62686 = 3.62686 \times 10^{-3}$ mho; while $R' = 1/g' = 275.7205$ ohms. $\rho' = 1000 \coth 2 - 275.7205 = 761.594$ ohms.

The equivalent Π of the line is a delta-connection of three resistances A B, A G″, B G″ (Figure 3), the two "pillars" or leaks AG″, BG″, being equal conductances of g'' mhos each, and the "architrave" AB being the line-resistance ρ''.

$$\rho'' = z \sinh \theta \qquad \text{ohms} \quad (19)$$

and

$$g'' = 1/R'' = y \tanh \frac{\theta}{2} \qquad \text{mhos}$$

$$= y \coth \theta - y'' = G_g - y'' \qquad \text{mhos,} \quad (20)$$

where $y'' = 1/g''$ is the architrave conductance, and $G_g = 1/R_g$ is the conductance to ground of the line at one end, when grounded at the other end.

Thus, for the line considered, $\rho'' = 1000 \sinh 2 = 3626.86$ ohms, and $g'' = 0.001 \coth 2 - 2.757204 \times 10^{-4} = 7.6159 \times 10^{-4}$ mho.

Single Line Corresponding to a Symmetrical T or Π.

Reciprocally, any star connection of three resistances AO, GO, BO (Figure 2), having two equal line-branches AO and OB of ρ' ohms, with a leak to ground of $R' = 1/g'$ ohms, corresponds to some smooth uniform line of angle,

$$\theta = 2 \sinh^{-1} \sqrt{\frac{\rho'}{2R'}} \qquad \text{hyps,} \quad (21)$$

and of surge-resistance,

$$z = \sqrt{\rho' (\rho' + 2R')} \qquad \text{ohms.} \quad (22)$$

Likewise, any delta-connection ABG″G″ (Figure 3) with two equal grounded leaks of resistance $R'' = 1/g''$ ohms, connected by an architrave of ρ'' ohms, corresponds to a smooth uniform line of angle,

$$\theta = 2 \tanh^{-1} \sqrt{\frac{\rho''}{2R'' \times \rho''}} \qquad \text{hyps,} \quad (23)$$

and of surge-resistance,

$$z = R'' \tanh \frac{\theta}{2} \qquad \text{ohms.} \quad (24)$$

Equivalent Circuits of Single Line in Terms of Resistances of Line Free and Grounded.

If the line be first freed and then grounded at one end, say B (Figure 1), and the resistance of the line be measured correctly at the other end in each case (R_f and R_g respectively), we have for the equivalent T of the line,

$$\rho' = R_f \left(1 - \sqrt{1 - \frac{R_g}{R_f}}\right) \qquad \text{ohms,} \quad (25)$$

$$R' = R_f \sqrt{1 - \frac{R_g}{R_f}} \qquad \text{ohms.} \quad (26)$$

Similarly, we have for the equivalent Π of the line,

$$\rho'' = R_g \Big/ \sqrt{1 - \frac{R_g}{R_f}} \qquad \text{ohms,} \quad (27)$$

$$R'' = R_f \left(1 + \sqrt{1 - \frac{R_g}{R_f}}\right) \qquad \text{ohms.} \quad (28)$$

From which

$$\frac{\rho' + R''}{2} = R_f \qquad \text{ohms.} \quad (29)$$

$$r/g = R''\rho' = R'\rho'' = R_f\,R_g = z^2 \qquad (\text{ohms})^2 \quad (30)$$

$$\theta = L\alpha = \tanh^{-1}\sqrt{\frac{R_g}{R_f}} \qquad \text{hyps.} \quad (31)$$

The last two formulas serve to evaluate z and θ for any single line, when the sending-end impedances of that line (R_f and R_g) have been correctly measured.

Looped or Metallic-Return Single Circuits.

If we consider single metallic circuits, like those of wire-telephony, or of single-phase power-transmission,

Let $r_{\shortparallel}$ = the linear resistance (ohms per loop km.).

$g_{\shortparallel}$ = the linear leakance (mhos per loop km.).

$l_{\shortparallel}$ = the linear inductance (henrys per loop km.).

$c_{\shortparallel}$ = the linear capacitance (farads per loop km.).

Then

$$\left.\begin{array}{ll} r_{\shortparallel} = 2\,r & \text{ohms per km.} \\ g_{\shortparallel} = g/2 & \text{mhos per km.} \\ l_{\shortparallel} = 2l & \text{henrys per km.} \\ c_{\shortparallel} = c/2 & \text{farads per km.} \end{array}\right\} \quad (32)$$

where r, g, l, and c are the corresponding linear constants per wire km. Substituting in equations (1), (2), and (3), we have

$$\alpha_{\shortparallel} = \alpha \qquad \text{hyps per loop km.,} \quad (33)$$

$$\theta_{\shortparallel} = \theta \qquad \text{hyps,} \quad (34)$$

and

$$z_{\shortparallel} = 2\,z \qquad \text{ohms.} \quad (35)$$

That is, the attenuation-constant, and the angle subtended by the looped line, are respectively identical with the attentuation-constant and angle subtended by one wire only operated to zero potential. The surge-impedance of the metallic circuit is double the surge-impedance of one wire to ground, or zero potential. The voltage impressed upon the loop is, however, double the voltage impressed on each wire singly

worked to zero-potential plane, so that the current-strength in the circuit is the same with either method of computation.

The above conditions are illustrated in Figure 4, where ABB′A repre-

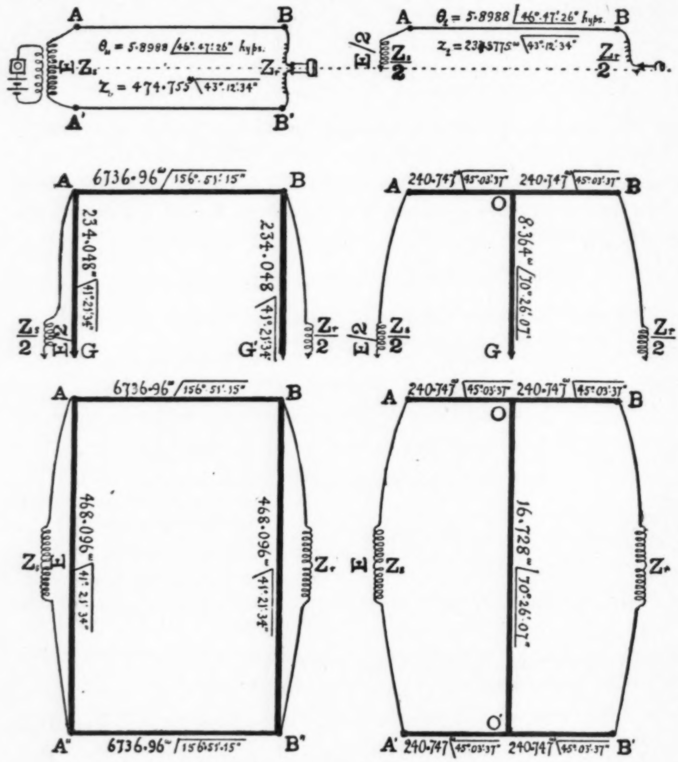

FIGURE 4. Equivalent circuits of lines with ground return and metallic return.

sents a simple metallic-return telephone circuit with a transmitter induction coil of impedance Z_s at A, and a receiver of impedance Z_r at B. One half of this circuit, with only one wire and ground return, is indicated at AB on the right hand. The length of the circuit has been taken as $L = 50$ km. (31.068 statute miles), and the following linear constants have been assumed:

$r_{\shortparallel} = 55.92$ ohms per loop km. (90 ohms per loop-mile); $g_{\shortparallel} = 0$
$l_{\shortparallel} = 0.70 \times 10^{-3}$ henry per loop km. (1.126 millihenry per loop-mile)
$c_{\shortparallel} = 0.049,7 \times 10^{-6}$ farad per loop km. (0.08×10^{-6} farad per loop-mile); values which correspond to

$$r = 27.96 \text{ ohms per wire km.}$$
$$l = 0.35 \times 10^{-3} \text{ henry per wire km.}$$
$$c = 0.099,4 \times 10^{-6} \text{ farad per wire km.}$$

Substituting the above values in (1), (2), and (3), we obtain at $\omega =$ 5,000 radians per second:

$a_{\shortparallel} = a = 0.117,976,6 \; \underline{/46^\circ\ 47'\ 26''}$ hyps per loop km., or per wire km.
$\theta_{\shortparallel} = \theta = 5.898,83 \; \underline{/46^\circ\ 47'\ 26''}$ hyps for both the double line and the single line.
$z_{\shortparallel} = 474.755 \; \overline{\backslash 43^\circ\ 12'\ 34''}$ ohms for the loop circuit.
$z = 237.377,5 \; \overline{\backslash 43^\circ\ 12'\ 34''}$ ohms for the single line.

The equivalent ⊓ and T of one wire are indicated at ABGG′ and AOBG in Figure 4. The architrave impedance AB is $6{,}736.96 \overline{/156^\circ\ 51'\ 15''}$ ohms, which is also the receiving-end impedance of each line, excluding the receiving instrument Z_r; because, if we ground the line at B, the current which will flow to ground at B will be the impressed potential at A divided by this architrave impedance.

The equivalent circuits of the loop line are indicated at ABB″A″ and AOBB′O′A′ (Figure 4). The former is a rectangle of impedances, and the latter an I of impedances. It will be seen that the rectangle ABB″A″ is merely a doublet of the single line ⊓, ABG′G; while the I, AOBB′O′A′ is merely a doublet of the single line T, AOBG. The receiving-end-impedance of the loop-circuit is evidently $2 \times 6{,}736.96 \overline{/156^\circ\ 51'\ 15''} = 13{,}473.92 \overline{/156^\circ\ 51'\ 15''}$ ohms, excluding the receiving instrument Z_r.

Since, then, the equivalent circuits of metallic-circuit or loop-lines are mere doublets of those for their component single wires, and the latter are easier to think about and discuss, we will confine our attention to the latter.

COMPOSITE LINES.

FIRST CASE. *Sections of the same Attenuation-Constant and of the same Surge-Impedance.*

If a line AB (Figure 5) of L_1 km. is connected to a line CD of L_2 km., and each has the same attenuation constant a, and the same surge-resistance z ohms (conditions which imply the same linear constants), the line angles will be $\theta_1 = L_1 a$ and $\theta_2 = L_2 a$ hyps respectively. Then, if we free the composite line at D, the resistance at A is

$$R_f = z \coth(\theta_1 + \theta_2) \qquad \text{ohms, (36)}$$

while, if the composite line be grounded at D, the resistance at A is

$$R_g = z \tanh(\theta_1 + \theta_2) \qquad \text{ohms. (37)}$$

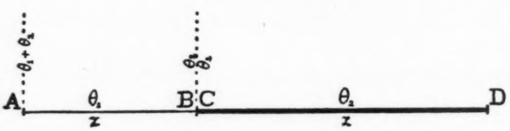

FIGURE 5. Composite line with sections of the same attenuation-constant and surge-resistance.

Reciprocally, freeing and grounding the composite line at A, we get resistances R_f and R_g at D, respectively the same as in (36) and (37).

It is evident, then, that the composite line differs in no way, except in length, from either of the component sections. The angle subtended by the whole line AD is the sum of the component section line-angles.

SECOND CASE. *Sections of different Attenuation-Constant but of the same Surge-Impedance.*

If a section CD (Figure 5) of L_2 km. be connected to a section AB of L_1 km., and their respective linear constants r_2, g_2, and r_1, g_1 are such that their attenuation constants a_1, a_2 differ; while their surge-resistances z are the same, we assign the angles subtended by the sections $\theta_1 = L_1 a_1$ and $\theta_2 = L_2 a_2$ hyps. The angle subtended by the whole line will then be $\theta_1 + \theta_2$, as in the preceding case. That is, except for a disproportionality between the section-angles and their line-lengths, two sections of different attenuation-constant, but of the same surge-resistance, connect together like two sections of one and

the same type of line. This is for the reason that in the unsteady state, or period of current building prior to the formation of the steady state here discussed, there is neither wave reflection nor discontinuity of wave propagation at the junction BC, when the surge resistance or impedance z is the same on each side thereof.

In order, however, to simplify the transition to complex cases later on, we may pause to consider the following case of two sections, with different a but the same z.

$$L_1 = 100 \text{ km.}, \; r_1 = 20 \text{ ohms/km.}, \; g_1 = 2 \times 10^{-5} \text{ mho/km.}$$
$$L_2 = 100 \text{ km.}, \; r_2 = 10 \text{ ohms/km.}, \; g_2 = 10^{-5} \text{ mho/km.}$$

Whence
$$a_1 = 0.02 \text{ hyp/km.}, \; z_1 = 1000 \text{ ohms};$$
$$a_2 = 0.01 \text{ hyp/km.}, \; z_2 = 1000 \text{ ohms.}$$

Merger Equivalent Circuits of Composite Lines.

Figure 6 shows the two lines at AB and CD respectively. It also shows the Π and T equivalent circuits of AB at A″B″G″G″ and A′OB′G′, likewise of CD at C″D″G″G″ and C′OD′G′. If we connect the sections together at BC, into a composite line AD, we virtually connect together some one pair of the combinations of equivalent circuits $\Pi_{AB}\Pi_{CD}$, $T_{AB}T_{CD}$, $\Pi_{AB}T_{CD}$, $T_{AB}\Pi_{CD}$. The first two combinations are shown at ABCDGGG and A′OBCOD′G′G′. If we merge together the two elements of any such pair by known formulas,[3] we arrive either at the equivalent Π, ADGG; or the equivalent T, AODG, of the composite line.

The equivalent Π or T of a composite line, computed by the merging of the Πs or Ts of the component sections, may be called the "merger Π" or "merger T" of the line, to distinguish them from the Π or T computed directly from the composite lines by the formulas to be presented later. The latter may be called, for distinction, the "hyperbolic Π" or T. For a given degree of precision, it will be found much easier to compute the hyperbolic Π or T of a composite line than to compute the merger Π or T. In all the examples given in this paper the equivalent Π and T of the various composite lines considered have both been derived hyperbolically, but have also been checked by the merging process.

[3] "The Equivalence of Triangles and Three-Pointed Stars in Conducting Networks," A. E. Kennelly, *Electrical World and Engineer,* Vol. 34, No. 12, Sept. 16, 1899, pp. 413–414.

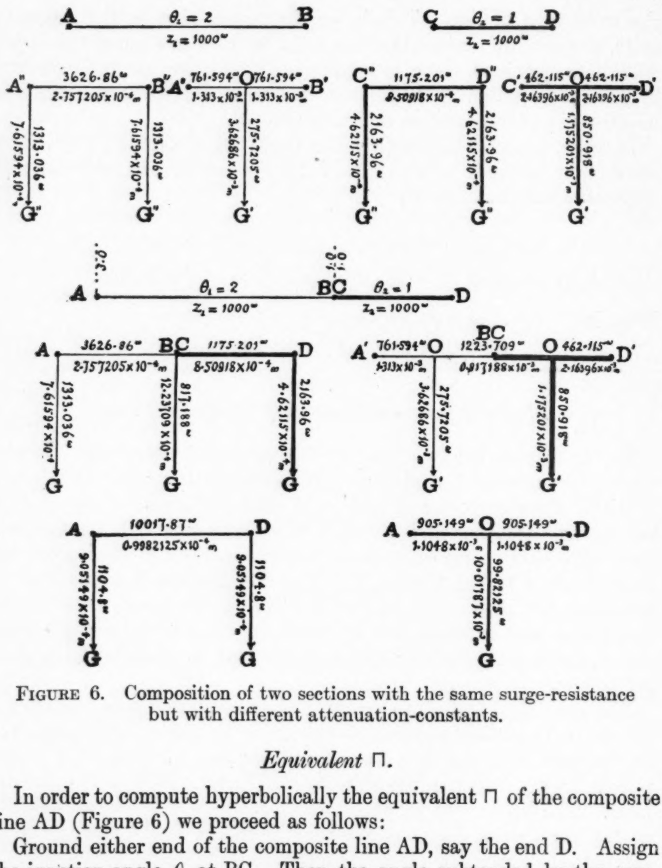

FIGURE 6. Composition of two sections with the same surge-resistance but with different attenuation-constants.

Equivalent ⊓.

In order to compute hyperbolically the equivalent ⊓ of the composite line AD (Figure 6) we proceed as follows:

Ground either end of the composite line AD, say the end D. Assign the junction-angle θ_2 at BC. Then the angle subtended by the composite line at A will be $\delta_A = \theta_1 + \theta_2$ hyps. The sending-end resistance of the composite line at A is, by (12) and (37),

$$R_{gA} = z_1 \tanh \delta_A \qquad \text{ohms} \quad (38)$$

$$= 1{,}000 \tanh 3 = 995.055 \text{ ohms.}$$

$$G_{gA} = 1/R_{gA} = y_1 \coth \delta_A \qquad \text{mhos} \quad (39)$$

$$= 0.001 \times \coth 3 = 10.049{,}7 \times 10^{-4} \text{ mho.}$$

Then the architrave resistance AD of the composite ⊓ will be:

$$\rho'' = z_1 \sinh \delta_A \qquad \text{ohms (40)}$$
$$= 1{,}000 \sinh 3 = 10{,}017.87 \text{ ohms.}$$
$$y'' = 1/\rho'' = 0.998{,}212{,}5 \times 10^{-4} \text{ mho.}$$

The conductance g''_A of the leak at A is, by (20),

$$g''_A = y_1 \coth \delta_A - y'' \qquad \text{mho (41)}$$
$$= 9.051{,}49 \times 10^{-4} \text{ mho.}$$

If we ground the composite line at A instead of at D, the angle subtended by the whole line at D will be $\delta_D = \theta_1 + \theta_2 = \delta_A$. The architrave resistance DA will be the same as that given in (40). The sending-end resistance R_{gD} and conductance G_{gD} will be identical with R_{gA} and G_{gA} respectively, by (38) and (39); so that the leak-conductance g''_D at D will be identical with g''_A by (41). This completes the hyperbolic ⊓, ADGG of the composite line.

Equivalent T.

To find the hyperbolic equivalent T of the composite line AD (Figure 6), free the line at one end, say D. Then the angle subtended by the line at A will be, as before, $\delta_A = \theta_1 + \theta_2$ hyps.

The sending-end resistance of the line at A will be, by (4),

$$R_{fA} = z \coth \delta_A \qquad \text{ohms (42)}$$
$$= 1{,}000 \coth 3 = 1{,}004.97 \text{ ohms.}$$

The conductance of the leak OG is, by (17),

$$g' = y \sinh \delta_A \qquad \text{mhos (43)}$$
$$= 0.001 \sinh 3 = 10.017{,}87 \times 10^{-3} \text{ mhos.}$$

and its resistance is

$$R' = 1/g' = 99.821{,}25 \text{ ohms.}$$

The resistance of the AO branch is, then, by (18),

$$\rho' = R_{fA} - R' \qquad \text{ohms (44)}$$
$$= 1{,}004.97 - 99.821 = 905.149 \text{ ohms.}$$

Similarly, if we free the composite line at A, instead of at D, the angle subtended by the line at D will be δ_D. As before, $\delta_D = \theta_2 + \theta_1 = \delta_A$

hyps. The sending-end resistance offered by the line at D will then be, by (4) and (42), identical with that found previously at A. The conductance of the leak will, by (17) and (43), be the same as that found from A. Finally, the resistance of the DO line-branch will, by (18) and (44), be identical with that of the AO branch (905.149^{ω}). This completes the T of the composite line.

We may infer from the above reasoning, and it may be readily demonstrated formally, that when a composite line is composed of sections differing in linear constants, but having the same surge-impedance, the angle subtended by the whole line is the same at either end, and whether the distant end be freed or grounded. Consequently the equivalent ⊓ and T of the composite line will be symmetrical. That is, the two leaks of the ⊓ are equal and the two line branches of the T are equal.

Conversely, it follows, from equations (21) to (24), that any composite line made up of sections differing in attenuation constant, but with the same surge-impedance, may be replaced by an equivalent single line of uniform attenuation and linear constants.

Third and General Case. Sections with Different Surge-Impedances.

Let a section AB of 100 km. (Figure 7) be connected to a section CD of 300 km., and let their respective linear constants be as follows:

$$r_1 = 20 \text{ ohms/km.}; \; g_1 = 20 \times 10^{-6} \text{ mho/km.}$$
$$r_2 = 10 \text{ ohms/km.}; \; g_2 = 2.5 \times 10^{-6} \text{ mho/km.}$$

from which

$$a_1 = 0.02 \text{ hyp/km.}; \; \theta_1 = 2 \text{ hyps}; \; z_1 = 1{,}000 \text{ ohms};$$
$$a_2 = 0.005 \text{ hyp/km.}; \; \theta_2 = 1.5 \text{ hyps}; \; z_2 = 2{,}000 \text{ ohms},$$

so that the surge-resistance of the two sections are unequal. It follows that the angle subtended by the composite line will differ at the two ends, and will also differ according to whether the distant end is freed or grounded.

Equivalent ⊓.

Let us ground the end A_2 of the composite line A_2D_2 (Figure 7). Then by formula (12), the sending-end resistance at B of the section BA grounded, will be

$$R_{gB} = z_1 \tanh \theta_1 \qquad \text{ohms} \quad (45)$$
$$= 1{,}000 \tanh 2.0 = 964.026{,}5 \text{ ohms}.$$

The angle of the section AB, at its end B, is $\delta_B = 2$ hyps. At the junction BC, however, the line-angle changes abruptly, owing to the change in surge-resistance, and at C, just across the junction it is

$$\delta_C = \tanh^{-1}\left(\frac{z_1}{z_2}\tanh\theta_1\right) = \tanh^{-1}\left(\frac{R_{gB}}{z_2}\right) \qquad \text{hyps. (46)}$$

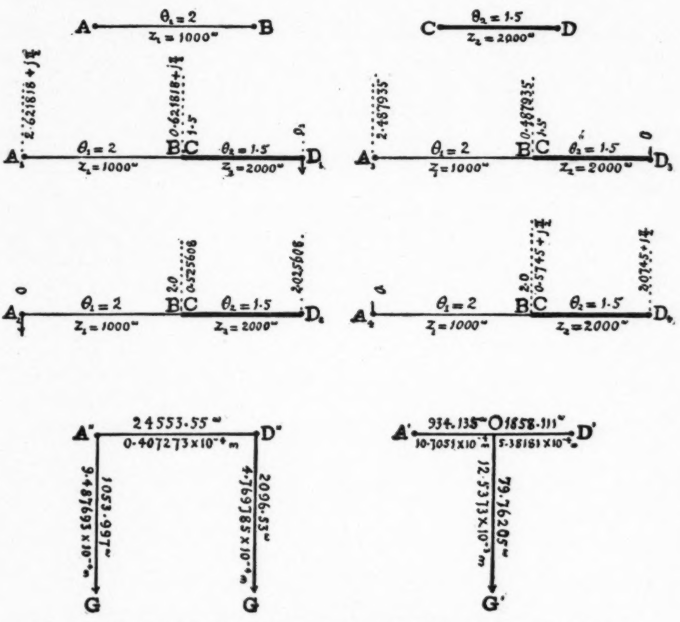

FIGURE 7. Composition of two sections of different surge-resistances and different attenuation-constants.

That is, the hyp-tangent of the new angle is the ratio of the sending-end resistance at B to the surge-resistance of the new section CD. In this case

$$\delta_C = \tanh^{-1}\left(\frac{964.026,5}{1000}\right) = \tanh^{-1} 0.964,026,5\,;$$

or, by tables of hyperbolic tangents, $\delta_c = 0.525,608$ hyp. We mark this angle opposite to C on the line A_2D_2 (Figure 7). The angle subtended at D_2 by the composite line is, therefore,

$$\delta_D = \theta_2 + \delta_C = 2.025{,}608 \text{ hyps.}$$

The sending-end resistance of the grounded composite line is then, at D_2, by (12), (37), (38), and (45),

$$R_{gD} = z_2 \tanh \delta_D \qquad \text{ohms (47)}$$
$$= 2{,}000 \tanh 2.025{,}608 = 1931.58 \text{ ohms,}$$

and the sending-end conductance,

$$G_{gD} = y_2 \coth \delta_D = 1/R_{gD} \qquad \text{mhos (48)}$$
$$= 0.000{,}517{,}71 \text{ mho.}$$

The formula for finding the architrave resistance of the equivalent ⊓ of the line AD is

$$\rho'' = z_2 \sinh \delta_D \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{ohms (49)}$$
$$= 2{,}000 \sinh 2.025{,}608 \times \frac{\cosh 2.0}{\cosh 0.525{,}608}$$
$$= 24{,}553.55 \text{ ohms}$$

and $$y'' = 1/\rho'' = 0.407{,}273 \times 10^{-4} \text{ mho.}$$

Formula (49) differs from the corresponding formula (40) of the preceding case by the application of the ratio $\frac{\cosh \delta_B}{\cosh \delta_C}$ or the ratio of the cosines of the line-angles across the junction BC.

The formula for finding the conductance of the leak at D is, as before (20) and (41),

$$g''_D = G_{gD} - y'' = 1/R_{gD} - y'' \qquad \text{mhos (50)}$$
$$= 4.769{,}785 \times 10^{-4} \text{ mho.}$$

In order to complete the equivalent ⊓ of the line AD hyperbolically, we must repeat the above process from the opposite end, by grounding the end D_1, as shown at A_1D_1 (Figure 7). The line angle at C is $\delta_C =$ 1.5 hyps. Across the junction BC this angle changes suddenly to

$$\delta_B = \tanh^{-1}\left(\frac{z_2 \tanh \theta_2}{z_1}\right) \qquad \text{hyps (51)}$$
$$= \tanh^{-1} 1.810{,}296.$$

This involves at first sight an impossible result; but in all cases of a hyperbolic tangent greater than unity, we may resort to the following formulas:

$$\left.\begin{aligned}\sinh\left(x \pm j\frac{\pi}{2}\right) &= \pm j\cosh x\\ \cosh\left(x \pm j\frac{\pi}{2}\right) &= \pm j\sinh x\\ \tanh\left(x \pm j\frac{\pi}{2}\right) &= \coth x\\ \coth\left(x \pm j\frac{\pi}{2}\right) &= \tanh x\end{aligned}\right\} \qquad \text{numeric. (52)}$$

We thus obtain

$$\begin{aligned}\delta_B - j\frac{\pi}{2} &= \coth^{-1}\left(\frac{z_2 \tanh 1.5}{z_1}\right) \qquad \text{hyps (53)}\\ &= \coth^{-1} 1.810{,}296\\ &= 0.621{,}818 \text{ hyp}\end{aligned}$$

and $$\delta_B = 0.621{,}818 + j\frac{\pi}{2} \text{ hyp.}$$

This difficulty with seemingly impossible antitangents or anticotangents is not encountered in the A. C. case.

We inscribe this value of δ_B opposite B on the line AD. The angle subtended by the whole line at A will then be

$$\theta_1 + \delta_B = \delta_A = 2.621{,}818 + j\frac{\pi}{2} \text{ hyps.}$$

The sending-end resistance of the grounded composite line is then at A_1, by (12), (37), (38), and (47),

$$\begin{aligned}R_{gA} &= z_1 \tanh \delta_A \qquad \text{ohms (54)}\\ &= 1{,}000 \tanh\left(2.621{,}818 + j\frac{\pi}{2}\right)\\ &= 1{,}000 \coth 2.621{,}618 = 1{,}010.64 \text{ ohms,}\end{aligned}$$

and the sending-end conductance, as in (48),

$$\begin{aligned}G_{gA} &= y_1 \coth \delta_A\\ &= y_1 \coth\left(2.621{,}818 + j\frac{\pi}{2}\right)\\ &= 0.001 \tanh 2.621{,}818 = 9.894{,}966 \times 10^{-4} \text{ mho.}\end{aligned}$$

The architrave resistance, as in (49), is

$$\rho'' = z_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \qquad \text{ohms (55)}$$
$$= 1{,}000 \cosh 2.621{,}818 \cdot \frac{\cosh 1.5}{\sinh 0.621{,}818}$$
$$= 24{,}553.55 \text{ ohms}$$

and $$y'' = 1/\rho'' = 0.407{,}273 \times 10^{-4} \text{ mho.}$$

The conductance of the ⊓ leak at A is, as in (50),

$$g''_A = G_{gA} - y''$$
$$= 9.487{,}693 \times 10^{-4} \text{ mho.}$$

Equivalent T.

To compute the equivalent T of the composite line AD (Figure 7), free the line at one end, say D_3, and find the sending-end resistance at C in this condition. It is, by (4), (36), and (42),

$$R_{fc} = z_2 \coth \theta_2$$
$$= 2{,}000 \coth 1.5 = 2{,}209.59 \text{ ohms.}$$

The line-angle changes abruptly at the junction BC from $\delta_C = 1.5$ to $\delta_B = 0.487{,}935$ hyp, by the condition

$$\delta_B = \coth^{-1}\left(\frac{z_2 \coth \theta_2}{z_1}\right) = \coth^{-1}\left(\frac{R_{fC}}{z_1}\right) \qquad \text{hyps (56)}$$
$$= \coth^{-1} 2.209{,}59 = 0.487{,}935 \text{ hyp.}$$

The line-angle at the end A_3 is thus $\theta_1 + \delta_B = 2.487{,}935$ hyps. The sending-end resistance at A_3 is finally, by (4),

$$R_{fA} = z_1 \coth \delta_A \qquad \text{ohms (57)}$$
$$= 1{,}000 \coth 2.487{,}935 = 1{,}013.897 \text{ ohms.}$$

The conductance of the leak OG′ is, by (43),

$$g' = y_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \qquad \text{mhos (58)}$$
$$= 0.001 \times \sinh 2.487{,}935 \times \frac{\cosh 1.5}{\cosh 0.487{,}935} = 12.537{,}3 \times 10^{-3} \text{ mho.}$$

The resistance of the leak OG′ is, therefore, $R' = 1/g' = 79.762$ ohms. The resistance of the AO branch is then, by (18) and (44),

$$\rho' = R_{fA} - R' \qquad \text{ohms (59)}$$
$$= 1{,}013.897 - 79.762 = 934.135 \text{ ohms.}$$

In order to complete the equivalent T of the line AD, we must repeat the above process from the opposite end, by freeing the end A, as shown at A_4D_4 (Figure 7). The line-angle at B is $\delta_B = 2.0$. Across the junction BC this angle changes suddenly to

$$\delta_C = \coth^{-1}\left(\frac{z_1 \coth \theta_1}{z_2}\right) \qquad \text{hyps (60)}$$
$$= \coth^{-1}\left(\frac{1037.315}{2{,}000}\right) = \coth^{-1} 0.518{,}657{,}5.$$

In order to avoid an impossible operation, apply formula (52)

$$\delta_C - j\frac{\pi}{2} = \tanh^{-1} 0.518{,}657{,}5 = 0.574{,}50 \text{ hyp}$$
$$\delta_C = 0.574{,}5 + j\frac{\pi}{2} \text{ hyps.}$$

The line-angle at the end D_4 is thus $\theta_2 + \delta_C = 2.074{,}5 + j\frac{\pi}{2}$ hyps.

The sending-end resistance at D_4 is finally, by (4) and (57),

$$R_{fD} = z_2 \coth \delta_D \qquad \text{ohms (61)}$$
$$= 2{,}000 \coth\left(2.074{,}5 + j\frac{\pi}{2}\right) = 2{,}000 \tanh 2.074{,}5 = 1{,}937.873 \text{ ohms.}$$

The conductance of the leak OG′ is, therefore, by (43) and (58),

$$g' = y_2 \sinh \delta_D \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{mhos (62)}$$
$$= 0.001 \sinh\left(2.074{,}5 + j\frac{\pi}{2}\right) \cdot \frac{\cosh 2.0}{\cosh\left(0.574{,}5 + j\frac{\pi}{2}\right)}$$
$$= 0.001 \cosh 2.074{,}5 \cdot \frac{\cosh 2.0}{\sinh 0.574{,}5} = 12.537{,}3 \times 10^{-3} \text{ mho.}$$

The resistance of the leak OG′ is, therefore, $R' = 1/g' = 79.762$ ohms. The resistance of the DO branch is then, by (18) and (59),

$$\rho' = R_{fD} - R' \qquad \text{ohms (63)}$$
$$= 1{,}937.873 - 79.762 = 1{,}858.111 \text{ ohms.}$$

This completes the T of the composite line.

It may be inferred from the preceding reasoning that for the case of a composite line of two sections with different surge-impedances, the receiving-end impedance of the line in the absence of receiving instruments, which is the architrave of the line-Π, has the same value from each end of the line. The leak of the composite line-T has also one and the same value, computed from either end. Both the Π and the T are, however, dissymmetrical. Each requires two separate computations and line-angle distributions, one from each end.

Summary of Two-Section Formulas.

If we expand formulas (40) and (49), we obtain for the architrave of the composite line Π

$$\rho'' = z_1 \sinh \theta_1 \cosh \theta_2 + z_2 \cosh \theta_1 \sinh \theta_2 \qquad \text{ohms} \quad (64)$$

$$= \frac{z_1 + z_2}{2} \sinh (\theta_1 + \theta_2) + \frac{z_1 - z_2}{2} \sinh (\theta_1 - \theta_2) \text{ ohms}^{4} \quad (65)$$

$$= z_1 \sinh \theta_1 \frac{\sinh \delta_D}{\sinh \delta_C} \qquad \text{ohms (line grounded at A)} \quad (66)$$

$$= z_2 \sinh \delta_D \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{ohms (line grounded at A)} \quad (67)$$

$$= z_2 \sinh \theta_2 \frac{\sinh \delta_A}{\sinh \delta_B} \qquad \text{ohms (line grounded at D)} \quad (68)$$

$$= z_1 \sinh \delta_A \frac{\cosh \delta_C}{\cosh \delta_B} \qquad \text{ohms (line grounded at D).} \quad (69)$$

Similarly, if we expand formulas (58) and (62), we obtain

$$g' = y_1 \sinh \theta_1 \cosh \theta_2 + y_2 \cosh \theta_1 \sinh \theta_2 \qquad \text{mhos} \quad (70)$$

$$= \frac{y_1 + y_2}{2} \sinh (\theta_1 + \theta_2) + \frac{y_1 - y_2}{2} \sinh (\theta_1 - \theta_2) \text{ mhos} \quad (71)$$

$$= y_1 \sinh \theta_1 \frac{\sinh \delta_D}{\sinh \delta_C} \qquad \text{mhos (line freed at A)} \quad (72)$$

$$= y_2 \sinh \delta_D \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{mhos (line freed at A)} \quad (73)$$

$$= y_2 \sinh \theta_2 \frac{\sinh \delta_A}{\sinh \delta_B} \qquad \text{mhos (line freed at D)} \quad (74)$$

$$= y_1 \sinh \delta_A \frac{\cosh \delta_C}{\cosh \delta_B} \qquad \text{mhos (line freed at D).} \quad (75)$$

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[4] Formulas (64) and (65) were first published as receiving-end impedances of a two-section composite line by Dr. G. di Pirro. See Bibliography.

Single Lines Equivalent to a Dissymmetrical Π *or* T.

It is evident that formulas (21) to (24) apply only to a symmetrical Π or T. Moreover, it may be seen that no single smooth and uniform line can correspond to a dissymmetrical Π or T. This means that, in general, no single smooth and uniform line can be the counterpart of a composite line having sections of different surge-resistance. But if we reduce a dissymmetrical Π to a symmetrical Π and a terminal leak, we may apply equations (23) and (24) to transform the symmetrical Π into an equivalent single line. It follows that any composite line may be resolved into one and only one uniform smooth line of the same length with a leak permanently applied to one end; or to an infinitude of such single uniform smooth lines having a leak at each end.

Similarly, the T of a composite line may be reduced to a symmetrical T plus a line-impedance at one end. By the use of equations (21) and (22), we may substitute a single smooth uniform line for the symmetrical T. Consequently, any composite line may be resolved into one and only one uniform smooth line of the same length with a line-impedance at one end; or, to an infinitude of such single uniform smooth lines having a line-impedance at each end.

COMPOSITE LINE WITH THREE SECTIONS OF DIFFERENT SURGE-IMPEDANCES.

A three-section composite line is indicated in Figure 8.

AB has a length L_1 of 100 km.
CD " " L_2 of 300 km.
EF " " L_3 of 50 km.

The respective linear constants are

$r_1 = 20$ ohms/km.; $r_2 = 10$ ohms/km.; $r_3 = 25$ ohms/km.
$g_1 = 20 \times 10^{-6}$ mho/km.; $g_2 = 2.5 \times 10^{-6}$ mho/km.; $g_3 = 4 \times 10^{-6}$ mho/km.
$\alpha_1 = 0.02$ hyp/km.; $\alpha_2 = 0.005$ hyp/km.; $\alpha_3 = 0.01$ hyp/km.
$\theta_1 = 2$ hyps; $\theta_2 = 1.5$ hyps; $\theta_3 = 0.5$ hyp.
$z_1 = 1000$ ohms; $z_2 = 2000$ ohms; $z_3 = 2500$ ohms.

Equivalent Π. *First Method.*

We proceed to compute the equivalent Π of the composite line AF in the same manner as in connection with Figure 7. Ground the end F_1 and develop the line-angles towards A_1. As before,

$$\delta_D = \tanh^{-1}\left(\frac{R_{gE}}{z_2}\right) \quad \text{and } \delta_B = \tanh^{-1}\left(\frac{R_{gC}}{z_1}\right) \quad \text{hyps.} \quad (76)$$

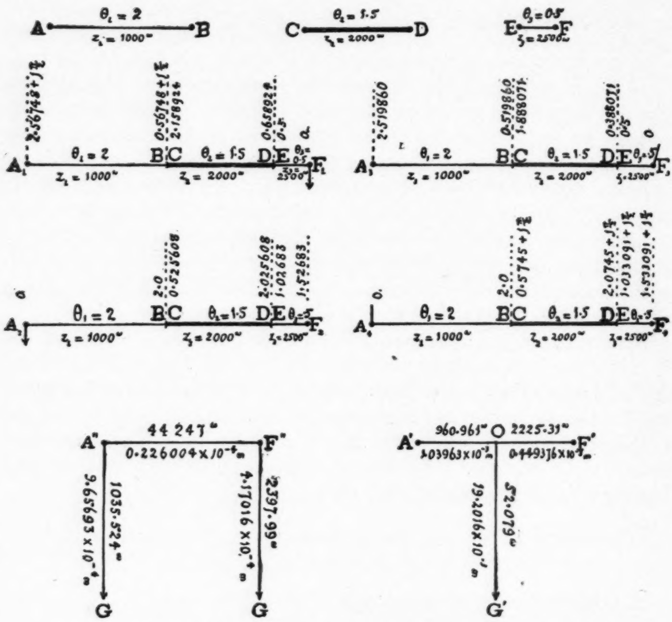

FIGURE 8. Composition of three sections of different surge-resistances.

The architrave resistance is then, following (49),

$$\rho'' = z_1 \sinh \delta_A \times \frac{\cosh \delta_C}{\cosh \delta_B} \times \frac{\cosh \delta_E}{\cosh \delta_D} \qquad \text{ohms} \quad (77)$$

$$= 1000 \cosh 2.567{,}48 \times \frac{\cosh 2.158{,}924}{\sinh 0.567{,}48} \times \frac{\cosh 0.5}{\cosh 0.658{,}923{,}6}$$

$$= 44{,}247 \ \text{ohms}.$$

The sending-end resistance at A is, as in (47),

$$R_{gA} = z_1 \tanh \delta_A \qquad \text{ohms} \quad (78)$$

$$= 1{,}000 \coth 2.567{,}48 = 1{,}011.84 \ \text{ohms}$$

The conductance of the ⊓ leak at A is, as in (50),

$$g''_A = 1/R_{gA} - 1/\rho'' \qquad \text{mhos.} \quad (79)$$

In order to complete the ⊓, we ground the line at A_2 (Figure 8), and develop the line-angles towards F_2. The architrave resistance is then

$$\rho'' = z_3 \sinh \delta_F \times \frac{\cosh \delta_D}{\cosh \delta_E} \times \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{ohms} \quad (80)$$

$$= 44{,}247 \text{ ohms.}$$

The sending-end resistance at F is

$$R_{gF} = z_3 \tanh \delta_F \qquad \text{ohms} \quad (81)$$

$$= 2{,}500 \tanh 1.526{,}83 = 2{,}274.71 \text{ ohms.}$$

Again,

$$g''_F = 1/R_{gF} - 1/\rho'' \qquad \text{mhos.} \quad (82)$$

Equivalent ⊓. *Second Method.*

An alternative method of arriving at the architrave resistance, which we may call the second method, is by following (66) and (68). Grounding at A_2, we have

$$\rho'' = z_1 \sinh \theta_1 \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_E} \qquad \text{ohms,} \quad (83)$$

and, grounding at F_1,

$$\rho'' = z_3 \sinh \theta_3 \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \qquad \text{ohms} \quad (84)$$

$$= 44{,}247 \text{ ohms.}$$

Equivalent T. *First Method.*

We proceed to compute the equivalent T of the composite line AF in the same manner as the T in Figure 7. Free the end F_3 and develop the line-angles towards A_3. As before,

$$\delta_D = \coth^{-1}\left(\frac{R_{fE}}{z_2}\right) \qquad \text{and } \delta_B = \coth^{-1}\left(\frac{R_{fC}}{z_1}\right) \qquad \text{hyps.} \quad (85)$$

The T leak conductance is then, following (58) and (75),

$$g' = y_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \qquad \text{mhos} \quad (86)$$

$$= 0.001 \sinh 2.519{,}86 \cdot \frac{\cosh 1.888{,}071}{\cosh 0.519{,}860} \cdot \frac{\cosh 0.5}{\cosh 0.388{,}071}$$

$$= 19.2016 \times 10^{-3} \text{ mho}$$

$$R' = 52.079 \text{ ohms.}$$

The sending-end resistance at A_3, as before, is

$$R_{fA} = z_1 \coth \delta_A \qquad \text{ohms (87)}$$
$$= 1,013.04 \text{ ohms.}$$

The AO line branch is therefore $R_{fA} - R' = 960.961$ ohms.

Repeating the process from A_4 towards F_4, we have for the T leak conductance, as in (80),

$$g' = y_3 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{mhos.(88)}$$

The sending-end resistance at F is likewise

$$R_{fF} = z_3 \coth \delta_F \qquad \text{ohms (89)}$$
$$= 2.500 \tanh 1.533,091 = 2,277.39 \text{ ohms,}$$

from which the resistance of the line branch FO follows.

Equivalent T. *Second Method.*

The second method of arriving at the T-leak conductance is by following (83) and (84). Freeing at A_4, we have

$$g' = y_1 \sinh \theta_1 \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_B} \qquad \text{mhos, (90)}$$

and freeing at F_3, after developing the line angles, we have

$$g' = y_3 \sinh \theta_3 \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \qquad \text{mhos. (91)}$$

Composite Line of n Sections.

To compute the equivalent ⊓ of a composite line of n successive sections, ground the line at the A end and develop the line-angles towards the opposite end, following the process of (76). Find the architrave impedance according to formula (80) or (83). This may be regarded as formula (19) modified by the application of $(n - 1)$ ratios of cosines in (80), or of $(n - 1)$ ratios of sines in (83). The opposite end leak admittance will then be the sending-end admittance minus the architrave admittance. The process must be repeated after grounding the line at the distant end and developing line-angles towards A.

To compute the equivalent T, free the line at the A end and develop the line-angles towards the opposite end, following the process of (85).

Find the ⊤-leak admittance by following formula (88) or (90). This may be regarded as formula (17) modified by the application of $n-1$ ratios of cosines in (88), or of $n-1$ ratios of sines in (90); that is, one such ratio for each junction. The opposite-end line-branch impedance will then be the sending-end impedance minus the leak impedance. The process must be repeated after freeing the line at the distant end and developing line-angles towards A.

One complete equivalent circuit, say the ⊓, of a composite line of n sections calls then for the determination $n-1$ line-angles first in one direction and then in the other. The formulas are well adapted to logarithmic computation. If, however, only the receiving-end impedance of the composite line is required, then we need only develop the line angles in one direction over the line so as to apply one of the architrave formulas, and neglect the pillars of the ⊓.

Loaded Composite Lines.

Definitions.

Loads in a line may be either *regular* or *casual.* Regular loads are such as are applied at regular intervals, in order to improve the current delivery on telephone lines. Casual loads are of an irregular or incidental character, such as might occur at section-junctions or at the ends of a composite line. In the former case they would be *intermediate* casual loads, and in the latter case, *terminal* casual loads. Only casual loads will be here discussed; because it is easy, with the aid of formulas already known, to substitute an equivalent smooth unloaded line for any uniformly loaded line.

Loads may also be divided into two classes; namely, (1) those applied in series with the line, or *impedance* loads, such as coils of impedance or resistance, and (2) those applied in derivation to the line, or *leak* loads.

Intermediate Impedance Loads.

The case of an intermediate impedance load, of 100 ohms, inserted at the junction BC in the composite line last considered, is presented in Figure 9. The system differs from that of Figure 8 only in the addition of this load.

Equivalent ⊓. *First Method.*

To compute the equivalent ⊓, A″F″GG (Figure 9), hyperbolically, ground the line at one end, say as at F_1, and develop the line-angles towards A_1. The only change in this process affected by the load is at the junction CB. The sending-end impedance at C is

$$R_{gC} = z_2 \tanh \delta_C \qquad \text{ohms} \quad (92)$$
$$= 2{,}000 \tanh 2.158{,}924 = 1{,}947.385 \text{ ohms}.$$

Consequently, if σ is the impedance of the load BC in ohms, the sending-end resistance at B is

$$R_{gB} = \sigma + z_2 \tanh \delta_C \qquad \text{ohms} \quad (93)$$
$$= 100 + 1{,}947.385 = 2{,}047.385 \text{ ohms},$$

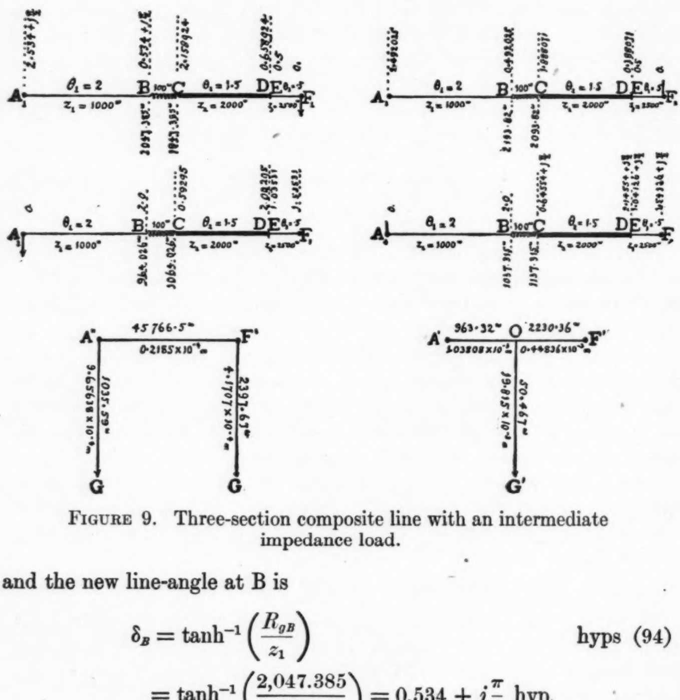

FIGURE 9. Three-section composite line with an intermediate impedance load.

and the new line-angle at B is

$$\delta_B = \tanh^{-1}\left(\frac{R_{gB}}{z_1}\right) \qquad \text{hyps} \quad (94)$$
$$= \tanh^{-1}\left(\frac{2{,}047.385}{1{,}000}\right) = 0.534 + j\frac{\pi}{2} \text{ hyp.}$$

Having established the angle of the whole line at A_1, the architrave impedance follows by formula (77) without further change. The A-leak is also obtained by formulas (78) and (79). In order to obtain the F-leak, and complete the ⊓, the line is grounded at the other end as at A_2 and the line-angles are developed towards F_2. At C, we have

$$\delta_C = \tanh^{-1}\left(\frac{100 + 964.026}{2{,}000}\right) = 0.592{,}95 \text{ hyp.}$$

Formulas (80), (81), and (82) then apply without change.

Equivalent Π. *Second Method.*

The alternative method for computing the architrave resistance of the line when grounded at A_2, and developed in angles, is

$$\rho'' = z_1 \sinh\theta_1 \cdot \frac{\sinh\delta_D}{\sinh\delta_C} \cdot \frac{\sinh\delta_F}{\sinh\delta_B} \cdot \frac{R_{gC}}{R_{gB}} \qquad \text{ohms,} \quad (95)$$

and when grounded at F_1 it is

$$\rho'' = z_3 \sinh\theta_3 \cdot \frac{\sinh\delta_C}{\sinh\delta_D} \cdot \frac{\sinh\delta_A}{\sinh\delta_B} \cdot \frac{R_{gB}}{R_{gC}} \qquad \text{ohms.} \quad (96)$$

That is, the effect of the load is to increase the architrave impedance in the ratio of the change of sending-end impedance across the load. In (95) this ratio is 1,064.026/964.026, and in (96) it is 2,047.385/1,947.385.

Equivalent T. *First Method.*

To compute the equivalent T of the loaded line in Figure 9, free the line at one end, as at F_3, and develop the line-angles towards A_3, as in (85). The only change effected by the load is in the angles at and beyond B. The sending-end impedance at C is

$$R_{fC} = z_2 \coth\delta_C \qquad \text{ohms} \quad (97)$$
$$= 2{,}000 \coth 1.888{,}071 = 2{,}093.82 \text{ ohms.}$$

The sending-end impedance at B is, therefore,

$$R_{fB} = \sigma + z_2 \coth\delta_C \qquad \text{ohms} \quad (98)$$
$$= 100 + 2{,}093.82 = 2{,}193.82 \qquad \text{ohms.} \quad (98)$$

The new line-angle at B is then

$$\delta_B = \coth^{-1}\left(\frac{R_{fB}}{z_1}\right) \qquad \text{hyps} \quad (99)$$
$$= \coth^{-1}\left(\frac{2{,}193.82}{1{,}000}\right) = 0.492{,}025 \text{ hyp.}$$

The T-leak admittance is now

$$g' = y_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \cdot \frac{G_{fC}}{G_{fB}} \quad \text{mhos} \quad (100)$$

$$= 0.001 \cdot \sinh 2.492,025 \cdot \frac{\cosh 1.888,071}{\cosh 0.492,025} \cdot \frac{\cosh 0.5}{\cosh 0.388,071} \cdot \frac{2,193.82}{2,093.82}$$

$$= 19.815 \times 10^{-3} \text{ mho.}$$

Formula (87) then applies without change.

Repeating the process from the opposite end of the line, as at A_4F_4, we have

$$g' = y_3 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_E} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \cdot \frac{G_{fB}}{G_{fC}} \quad \text{mhos} \quad (101)$$

$$= 19.815 \times 10^{-3} \text{ mho.}$$

Formula (89) then applies without change.

The effect of the load on the T-leak admittance formulas (86) and (88) is to alter them in the ratio of the impedances or admittances across the load, applying the said ratio in such a manner as to increase the result in the direct-current case.

Equivalent T. *Second Method.*

Formulas (90) and (91) of the alternative method are not altered by an intermediate impedance load, after the line-angles have been properly assigned.

Equivalence of Alternating-Current Transformers to Impedance Loads.

It may be observed that since the insertion of a transformer into a circuit, as, for example, the insertion of a "repeating-coil" into a telephone circuit, is theoretically equivalent to the insertion of impedance into the circuit without rupture of continuity, all cases of line transformers are capable of being dealt with by substituting for such transformers their equivalent intermediate impedance loads.[5]

Terminal Impedance Loads.

A terminal impedance load is likely to present itself in a composite line, owing to the presence of terminal apparatus. The architrave impedance of a composite line ⊓, computed without any terminal load, can only represent the receiving-end impedance of the line when the

[5] "On the Predetermination of the Regulation of Alternating-Current Transformers," A. E. Kennelly, *Electrical World and Engineer*, Sept. 2, 1899, Vol. 34, p. 343.

receiving apparatus is short-circuited. For example, in the case of Figure 4, if we short circuit the receiver Z_r the receiving-end impedance of each line is $6{,}736.96\overline{/156^\circ\ 51'\ 15''}$ ohms. With the receiver Z_r inserted, the receiving-end impedance is considerably changed, and this is the condition met with in practice. By applying half the im-

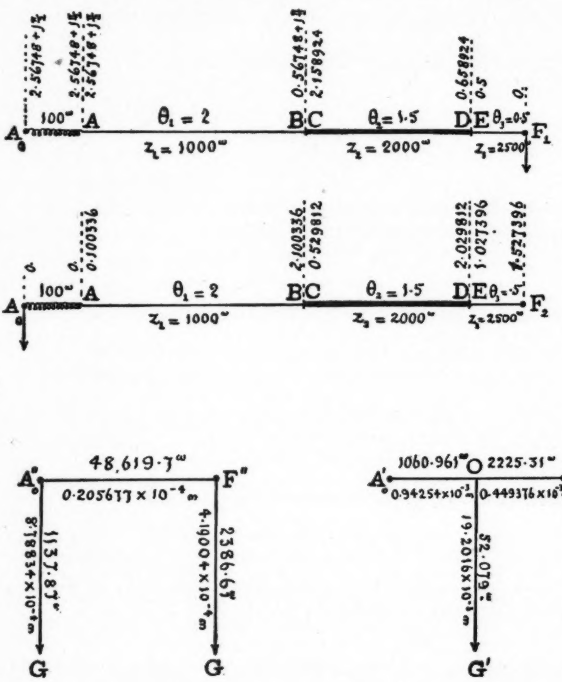

FIGURE 10. Three-section composite line with a terminal impedance load.

pedance of the receiver as a terminal load to the line, the architrave of the new equivalent ⊓ gives the receiving-end impedance with the receiver included. If this is the result sought, it becomes unnecessary to compute the values of the leaks of this ⊓.

Equivalent ⊓. *First Method.*

Figure 10 represents the three-section composite line of Figure 8, with a terminal impedance of 100 ohms applied at A. To compute

the equivalent ⊓ of the loaded line, ground F, as at F_1. Develop the line-angles towards A in the usual way. No change from the corresponding conditions of Figure 8 occurs until after we have reached δ_A. We then have

$$R_{gA} = z_1 \tanh \delta_A \text{ ohms}$$
$$= 1{,}000 \coth 2.567{,}48 = 1{,}011.607 \text{ ohms,}$$

and if σ be the impedance of the terminal load at A_0,

$$R_{gA0} = \sigma + z_1 \tanh \delta_A \qquad \text{ohms} \quad (103)$$
$$= z_0 \tanh \delta_A \qquad \text{ohms} \quad (104)$$
$$= 1{,}111.84 \text{ ohms,}$$

where z_0 is the *apparent surge-impedance* of the line at A_0; or

$$z_0 = z_1 + \sigma \coth \delta_A \qquad \text{ohms} \quad (105)$$
$$= R_{gA0}/\tanh \delta_A \qquad \text{ohms} \quad (106)$$
$$= 1{,}098.829 \text{ ohms.}$$

The architrave resistance is then, following (77),

$$\rho'' = z_0 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_D} \qquad \text{ohms} \quad (107)$$
$$= 48{,}619.7 \text{ ohms.}$$

The A-leak of the ⊓, as in the case of Figure (8), is

$$g''_A = 1/R_{gA0} - 1/\rho'' \qquad \text{mhos.} \quad (108)$$

To complete the ⊓, we ground the loaded line at A, as at A_0F_2, and develop the line-angles towards F, commencing with

$$\delta_A = \tanh^{-1}\left(\frac{\sigma}{z_1}\right) \qquad \text{hyps} \quad (109)$$
$$= \tanh^{-1}\left(\frac{100}{1000}\right) = 0.100{,}336 \text{ hyp.}$$

The architrave impedance is then

$$\rho'' = z_3 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \cdot \frac{\cosh 0}{\cosh \delta_A} \qquad \text{ohms} \quad (110)$$
$$= 48{,}619.7 \text{ ohms.}$$

The F-leak is then computed as in (82).

Equivalent Π. *Second Method.*

The alternative method gives

$$\rho'' = z_1 \cdot \frac{\sinh \delta_B}{\cosh \delta_A} \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_E} \quad \text{ohms,} \quad (111)$$

with the line grounded at A_0, and

$$\rho'' = z_3 \sinh \theta_3 \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \cdot \frac{R_{gA0}}{R_{gA}} \quad \text{ohms,} \quad (112)$$

with the line grounded at F.

Equivalent T.

To arrive at the equivalent T of a composite line loaded with a terminal impedance, all that is necessary is to find the T of the same line unloaded, by preceding formulas, and then to add the terminal impedance to the proper line-branch of this T.

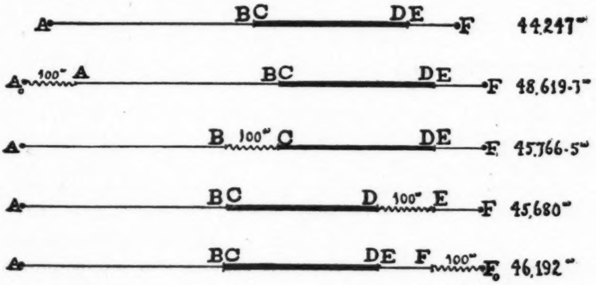

FIGURE 11. Diagram showing the influence of the location of an impedance load on the receiving-end resistance of a three-section composite line.

INFLUENCE OF LOCATION OF AN IMPEDANCE LOAD ON THE RECEIVING-END IMPEDANCE OF A COMPOSITE LINE.

It has been shown in a preceding paper that if a single smooth uniform line is terminally loaded with a given impedance, the change in the receiving-end impedance due to the load is the same, whichever end of the line the load may be applied to; *i. e.*, whether the load is applied at the sending or at the receiving end. In the case of a composite line, however, this proposition generally fails. The effect of a resistance coil of 100 ohms on the receiving-end resistance of the three-section composite line above discussed, is shown in Figure 11. With-

out the load, the receiving-end resistance of the line, or the architrave of its equivalent ⊓, is, by Figure 8, 44,247 ohms. If the load is added at the A end of the line, the receiving-end resistance becomes 48,619.7 ohms; but if added at the F end, it is only 46,192. When the same coil is inserted as an intermediate load, its influence on the receiving-end resistance is not so great. In A. C. composite lines, the opportunities for such variations are more marked. In all cases, however, the application of a terminal impedance σ to a line (single or composite), increases the receiving-end or architrave impedance of that line in the ratio $\frac{R_g + \sigma}{R_g}$; where R_g is the sending-end-impedance of the line at the loaded end before the load is applied. This is true whether the loaded end is made the sending or receiving end of the circuit. For single lines, R_g has the same value at either end, and therefore the ratio of increase in receiving-end impedance is the same at whichever end of a single line the load σ is applied; whereas, for composite lines, we have seen that R_g is different, in general, at the two ends.

Intermediate Leak Loads.

Equivalent ⊓. *First Method.*

Suppose a leak load to be applied at a junction between sections such as at DE (Figure 12). We proceed to compute the equivalent ⊓ of the loaded composite line by grounding one end, as at F_1. We develop the line-angles towards A_1. On arriving at E we have $R_{gE} = z_3 \tanh \theta_3 = 1{,}155.292$ ohms. Hence $G_{gE} = 1/R_{gE} = 8.655{,}82 \times 10^{-4}$ mho. To this sending-end admittance we add the admittance γ of the leak; so that the sending-end admittance at D, including the leak, is

$$G_{gD} = \gamma + G_{gE} \qquad \text{mhos} \quad (113)$$
$$= 13.655{,}82 \times 10^{-4} \text{ mho.}$$

Consequently the sending-end resistance at D, including the leak, is

$$R_{gD} = 1/G_{gD} \qquad \text{ohms,} \quad (114)$$
$$= 732.289 \text{ ohms.}$$

The line-angle at D is then

$$\delta_D = \tanh^{-1}\left(\frac{R_{gD}}{z_2}\right) \qquad \text{hyps} \quad (115)$$
$$= 0.383{,}964 \text{ hyp.}$$

The remaining line-angles are found in the regular way.

The architrave impedance is then

$$\rho'' = z_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \cdot \frac{R_{gE}}{R_{gD}} \qquad \text{ohms (116)}$$

$$= 60{,}240 \text{ ohms.}$$

The A-leak is computed regularly from (78) and (79).

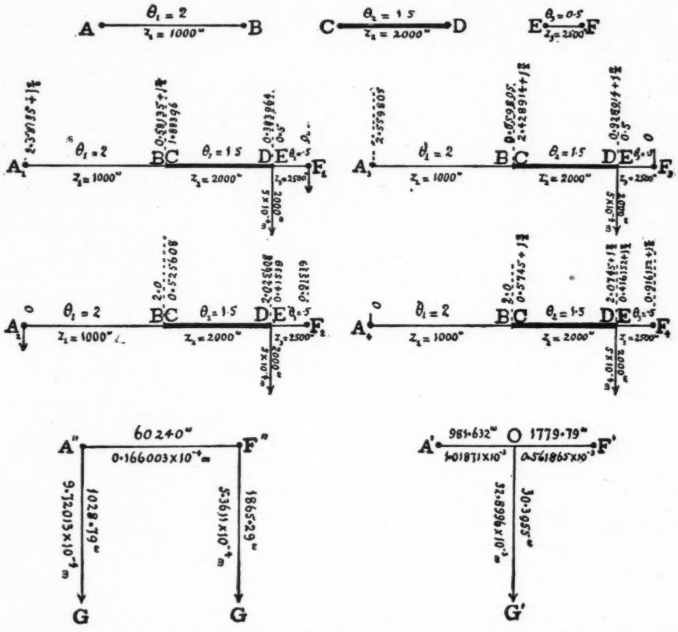

FIGURE 12. Composite line of three sections with intermediate leak load.

To complete the ⊓, ground the line at the opposite end, as at A_2, and develop the line-angles towards F_2, in the same manner as above. The architrave impedance is then

$$\rho'' = z_3 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_E} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \cdot \frac{R_{gD}}{R_{gE}} \qquad \text{ohms (117)}$$

$$= 60{,}240 \text{ ohms.}$$

The F-leak is computed regularly from (81) and (82).

Equivalent ⊓. *Second Method.*

In the alternative method we have the regular formulas (83) and (84), unchanged by the intermediate leak load.

Equivalent ⊤. *First Method.*

To complete the equivalent ⊤, free one end of the line, say F, as at F_3 (Figure 12), and develop the line-angles towards A_3. At the loaded junction DE we have

$$G_{fD} = \gamma + G_{fE} \qquad \text{mhos, (118)}$$
$$= 6.848{,}47 \times 10^{-4} \text{ mho,}$$

and, following (114) and (115),

$$\delta_D = \coth^{-1}\left(\frac{R_{fD}}{z_2}\right) \qquad \text{hyps, (119)}$$
$$= 0.928{,}914 + j\frac{\pi}{2} \text{ hyp.}$$

The remaining line-angles follow regularly. The ⊤-leak conductance also follows from (86) without change, and the line-branch AO is computed regularly by (18), (57), (59), and (87).

To complete the ⊤, free the other end of the line as at A_4, and proceed, as above, to develop the line-angles towards F_4. The ⊤-admittance must then conform to (88), and the line-branch impedance FO to (89).

Equivalent ⊤. *Second Method.*

The alternative method of arriving at the ⊤-leak admittance is by following (83) and (84). Freeing at A_4 (Figure 12), we have

$$g' = y_1 \sinh\theta_1 \cdot \frac{\sinh\delta_D}{\sinh\delta_C} \cdot \frac{\sinh\delta_F}{\sinh\delta_E} \cdot \frac{G_{fE}}{G_{fD}} \qquad \text{mhos, (120)}$$

and similarly, freeing at F_4, we have

$$g' = y_3 \sinh\theta_3 \cdot \frac{\sinh\delta_C}{\sinh\delta_D} \cdot \frac{\sinh\delta_A}{\sinh\delta_B} \cdot \frac{G_{fD}}{G_{fE}} \qquad \text{mhos. (121)}$$

Terminal Leak Loads.

Equivalent ⊓.

To arrive at the equivalent ⊓ of a composite line loaded with a terminal leak, such as that represented at AF in Figure 13, first compute

the equivalent Π of the same line unloaded, by preceding formulas, and then to the proper leak of the Π add the terminal load leak, numerically in the D. C. case, vectorially in an A. C. case.

Equivalent T. *First Method.*

To compute the equivalent T, free one end of the line, say F, as at F_3 (Figure 13), and develop the line-angles towards A. We commence with

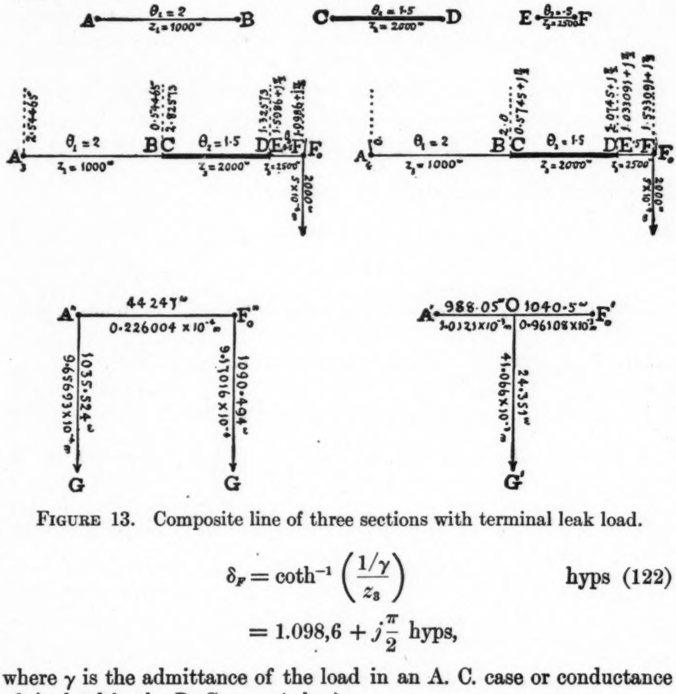

FIGURE 13. Composite line of three sections with terminal leak load.

$$\delta_F = \coth^{-1}\left(\frac{1/\gamma}{z_3}\right) \qquad \text{hyps} \quad (122)$$

$$= 1.098{,}6 + j\frac{\pi}{2} \text{ hyps,}$$

where γ is the admittance of the load in an A. C. case or conductance of the load in the D. C. case (mhos).

The T-leak admittance is then

$$g' = y_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_D} \cdot \frac{\cosh 0}{\cosh \delta_F} \qquad \text{mhos} \quad (123)$$

$$= 41.066 \times 10^{-8} \text{ mho,}$$

and the line-branch impedance AO follows at once from (87).

To complete the T, the line is freed at A, as at A_4 (Figure 13), and the line-angles are developed toward F. We then have for the sending-end admittance at F,

$$G_{fF} = y_3 \tanh \delta_F \qquad \text{mhos.} \quad (124)$$

The sending-end conductance at F_0, including the leak admittance

$$G_{fF_0} = \gamma + y_3 \tanh \delta_F \qquad \text{mhos.} \quad (125)$$

The apparent surge-admittance y_0 at F_0 is defined by the condition,

$$G_{fF_0} = y_0 \tanh \delta_F \qquad \text{mhos,} \quad (126)$$

whence

$$y_0 = y_3 + \gamma \coth \delta_F \qquad \text{mhos.} \quad (127)$$

The T-leak admittance will then conform to

$$g' = y_0 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \qquad \text{mhos} \quad (128)$$

$$= y_3 \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_B} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \cdot \frac{G_{fF_0}}{G_{fF}} \qquad \text{mhos,} \quad (129)$$

and the line-branch impedance FO follows at once from (89).

Equivalent T. *Second Method.*

By the alternative method, the T-leak admittance, when the line is freed at A, is

$$g' = y_1 \sinh \theta_1 \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_B} \cdot \frac{G_{fF_0}}{G_{fF}} \qquad \text{mhos} \quad (130)$$

$$= 41.066 \times 10^{-3} \text{ mho.}$$

Similarly, when the line is freed at F_0 (Figure 13), and the corresponding line-angles are set,

$$g' = y_3 \frac{\sinh \delta_E}{\cosh \delta_F} \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \qquad \text{mhos.} \quad (131)$$

The line-branch impedances are determined in the regular way.

Résumé of Rules applying to Casual Loads in Composite Lines.

In the accompanying Table the changes effected by loads in the formulas for ρ'' and g' are collected together as an aid to computation.

It will be seen that there is a certain symmetry in these changes that assists their application. Moreover, it is possible, after consulting the Table, to select in some particular case a method which avoids additional computation. Thus, in dealing with an intermediate leak, the first method calls for the application of the impedance ratio across the leak, to the formula for ρ''; whereas the second method calls for no change in its formula.

TABLE SHOWING CHANGES MADE BY CASUAL LOADS IN THE COMPOSITE-LINE FORMULAS FOR THE EQUIVALENT Π-ARCHITRAVE AND EQUIVALENT T-LEAK.

Nature of Load.	Change in the Formula for ρ''.		Change in the Formula for g'.	
	By First Method.	By Second Method.	By First Method.	By Second Method.
Intermediate impedance	None	R_{gM}/R_{gN}	G_{gM}/G_{gN}	None
Intermediate leak	R_{gM}/R_{gN}	None	None	G_{gM}/G_{gN}
Terminal impedance:				
At far end	$\cosh 0/\cosh \delta_N$	Subst.: $\frac{\sinh \delta_{N-1}}{\cosh \delta_{N0}}$ for $\sinh \theta_N$		
At near end	R_{gA0}/R_{gA} or subst. z_0 for z_1	R_{gA0}/R_{gA}		
Terminal leak:				
At far end			$\cosh 0/\cosh \delta_N$	Subst.: $\frac{\sinh \delta_{N-1}}{\cosh \delta_{N0}}$ for $\sinh \theta_N$
At near end			G_{fA0}/G_{fA} or subst. y_0 for y_1	G_{fA0}/G_{fA}

The ratios R_{gM}/R_{gN} and G_{gM}/G_{gN} denote respectively the ratios of sending-end impedance and sending-end admittance across the load, the ratio being taken in each case such that in the D. C. case it is greater than unity.

The far end is in all cases the end of the composite line which is to be considered as freed or grounded for the purposes of the computation, and the near end is the opposite end, or the end towards which the line-angles are developed.

It has been assumed for the purposes of the Table that the A end of the line happens to be the near end in all cases, and the N end the far end.

Plurality of Loads.

When several casual loads exist simultaneously in a composite line, each requires to be considered separately in the formulas for ρ'' and g', although no special treatment is involved thereby in computing g'' or ρ'. A particular case of this kind is shown in Figure 14, where the

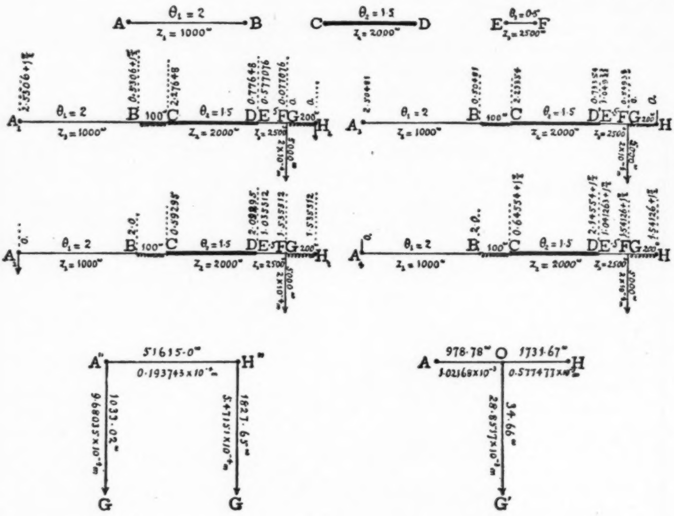

Figure 14. Composite line of three sections with two terminal and one intermediate load.

composite line of Figure 8 is loaded with an intermediate resistance of 100 ohms at the junction BC, a terminal resistance of 200 ohms at F and also with a terminal leak of 5000 ohms at F. The presence of the terminal resistance GH, however, converts the leak into an intermediate leak so far as concerns the process of computation.

Equivalent ⊓. *First Method.*

In order to compute the equivalent ⊓, ground the line at one end, as at A_2 (Figure 14), and develop the line-angles towards H by preceding formulas. Referring to the Table, we have (a) one intermediate impedance at BC; (b) one intermediate leak at FG, and (c) one terminal impedance at the near end H, the distant end being grounded.

Consequently, so far as concerns the first method, we should make no change in the formula for ρ'' on account of (a), but introduce the ratio R_{gF}/R_{gG} for (b) and substitute z_0 for z_3 on account of (c). Consequently, following (77) with these changes,

$$\rho'' = z_0 \sinh \delta_H \cdot \frac{\cosh \delta_D}{\cosh \delta_E} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \cdot \frac{R_{gF}}{R_{gG}} \quad \text{ohms} \quad (132)$$

$$= 1{,}936.87 \sinh 1.535{,}312 \cdot \frac{\cosh 2.092{,}95}{\cosh 1.035{,}31} \cdot \frac{\cosh 2.0}{\cosh 0.592{,}95} \cdot \frac{1809.74}{1565.14}$$

$$= 51{,}615 \text{ ohms.}$$

The H-leak is then found in the usual way.

Equivalent Π. *Second Method.*

Similarly, by reference to the Table, for changes in the ρ'' formula under the second method, we should introduce the ratio R_{gC}/R_{gB} for (a), make no change for (b), but introduce the ratio R_{gH}/R_{gG} for (c). Consequently, following (83) with these changes,

$$\rho'' = z_1 \sinh \theta_1 \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_E} \cdot \frac{R_{gC}}{R_{gB}} \cdot \frac{R_{gH}}{R_{gG}} \quad \text{ohms} \quad (133)$$

$$= 51{,}615 \text{ ohms.}$$

The A-leak is then computed in the regular manner.

If now we ground the line at the H-end, we obtain similarly, by the first method,

$$\rho'' = z_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \cdot \frac{\cosh 0}{\cosh \delta_F} \cdot \frac{R_{gG}}{R_{gF}} \quad \text{ohms} \quad (134)$$

$$= 51{,}615 \text{ ohms,}$$

and by the second method,

$$\rho'' = z_3 \frac{\sinh \delta_E}{\sinh \delta_F} \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \cdot \frac{R_{gB}}{R_{gC}} \cdot \frac{G_{gF}}{G_{gG}} \quad \text{ohms.} \quad (135)$$

Equivalent T. *First Method.*

Freeing the line at H, as at A_3H (Figure 14), we have

$$g' = y_1 \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \cdot \frac{\cosh \delta_G}{\cosh \delta_F} \cdot \frac{G_{fC}}{G_{fB}} \quad \text{mhos} \quad (136)$$

$$= 0.001 \cdot \sinh 2.504{,}81 \cdot \frac{\cosh 2.233{,}54}{\cosh 0.504{,}81} \cdot \frac{\cosh 1.049{,}31}{\cosh 0.733{,}54} \cdot \frac{1}{\cosh 0.549{,}31} \cdot \frac{2{,}146.46}{2{,}046.46}$$

$$= 28.851{,}7 \times 10^{-3} \text{ mho,}$$

and freeing the line at A, as at A_4, we have

$$g' = y_0 \sinh \delta_H \cdot \frac{\cosh \delta_D}{\cosh \delta_E} \cdot \frac{\cosh \delta_E}{\cosh \delta_C} \cdot \frac{G_{fB}}{G_{fC}} \cdot \frac{G_{fF}}{G_{fG}} \quad \text{mhos.} \quad (137)$$

Equivalent T. *Second Method.*

Freeing at H, we have

$$g' = y_3 \frac{\sinh \delta_E}{\cosh \delta_F} \cdot \frac{\sinh \delta_C}{\sinh \delta_D} \cdot \frac{\sinh \delta_A}{\sinh \delta_B} \quad \text{mhos,} \quad (138)$$

and freeing at A,

$$g' = y_1 \sinh \theta_1 \cdot \frac{\sinh \delta_D}{\sinh \delta_C} \cdot \frac{\sinh \delta_F}{\sinh \delta_E} \cdot \frac{G_{fG}}{G_{fF}} \quad \text{mhos.} \quad (139)$$

Methods of Computation adapted to Alternating-Current Cases.

There is especial need for brief methods of computation when A. C. cases are dealt with,[6] owing to the complexity of the vector arithmetic. In practice, the degree of precision desired will usually be much lower than that aimed at in the arithmetical examples of this paper, where the numerical values have been carried to six significant digits. Graphical methods may be frequently used with advantage, especially in the vector addition of complex hyperbolic angles. Traverse Tables as used by navigators may also be used with advantage for the resolution of vectors into complex quantities.

The following formulas are also useful:

$$\cosh (p \pm jq) = \sqrt{\cosh^2 p - \sin^2 q}\, \underline{/ \pm \tan^{-1}(\tanh p \cdot \tan q)} \quad (140)$$

$$\sinh (p \pm jq) = \sqrt{\sinh^2 p + \sin^2 q}\, \underline{/ \pm \tan^{-1}(\coth p \cdot \tan q)} \quad (141)$$

$$\tanh (p \pm jq) = \frac{\sinh 2p}{\cosh 2p + \cos 2q} \pm j \frac{\sin 2q}{\cosh 2p + \cos 2q} \quad (142)$$

$$\tanh^{-1}(p \pm jq) = \tfrac{1}{2} \log_e \sqrt{\frac{(1+p)^2+q^2}{(1-p)^2+q^2}} + j \left\{ \frac{\pi - \tan^{-1}\left(\frac{1+p}{\pm q}\right) - \tan^{-1}\left(\frac{1-p}{\pm q}\right)}{2} \right\} \quad (143)$$

[6] A table of hyperbolic tangents of a vector variable or of tanh $r\underline{/\theta}$, is being prepared by the writer for values of r between 0 and 6, by steps of 0.1 or less; and for virtually all angles θ, by steps of one degree.

Conclusions.

Any composite line of any number of sections, with or without loads of any kind, operated in the steady state either by a direct current, or by an alternating current of one frequency, has the same receiving-end impedance from each end; so that, if one volt be applied to each end in turn, the current strength received at the other end will be the same.[7]

The equivalent circuits of such lines may always be computed either for the D. C. or A. C. case by the formulas given in this paper. That is, any such line may always be replaced by one delta connection or by one star connection of impedance, without disturbing the electrical conditions outside of the line.

Notation Employed

$a, a_{//}, a_1, a_2, a_3$	attenuation-constants of a single line, of a loop-line, and of different sections of a composite line (hyps. per km.).
$c, c_{//}, c_1, c_2, c_3$	linear capacitance of single line, loop-line, and sections (farads/km.).
$\delta, \delta_A, \delta_B, \ldots$	the hyp. angles of points on a line (hyps).
$G, G_g, G_{gA}, \ldots G_f, G_{fA}, \ldots$	the sending-end admittance (D. C. conductance) of a line, the admittance beyond a point on the same, when the far end is grounded, and when the far end is free (mhos).
$g, g_{//}, g_1, g_2, g_3$	linear conductance of single line, loop-line, and sections (mhos/km.).
$g' = 1/R'$	conductance of leak of a T (mhos).
$g'' = 1/R''$	conductance of leak of a ⊓ (mhos).
γ	conductance of a leak load (mhos).
i, i_A, i_P	current strength, at the sending-end, and at a point on the line (amperes).
j	$\sqrt{-1}$
$l, l_{//}, l_1, l_2, l_3$	linear inductance of single line, loop-line, and sections (henrys/km.).
L, L_1, L_2, L_3	length of a line and of sections (km.).

[7] An exception should be noted in the case of any part of the composite line not obeying Ohm's law, as, for example, a fault in the insulation; so that the current through the fault is not proportional to the potential at the same.

l' . . . . distance of a point on a line from its far end (km.).

n . . . . frequency of single A. C. (cycles per second).

ω . . . . angular velocity of A. C. (radians per second).

p, q . . . . cartesian coördinates of a point in a plane.

r, r_{ll}, r_1, r_2, r_3 . . . . . linear resistance of a single line, loop-line, and sections (ohms/km.).

$R, R_g, R_{gA} \ldots, R_f, R_{fA} \ldots$ resistance of a line beyond a point on the same, the resistance when the far end is grounded, and when the far end is free, A. C. impedance (ohms).

$R' = 1/g'$ . . . . . . resistance (A. C. impedance) of leak of a $\top$ (ohms).

$R'' = 1/g''$ . . . . . . resistance (A. C. impedance) of leak of a $\sqcap$ (ohms).

$\rho' = 1/y'$ . . . . . . resistance (A. C. impedance) of line-branch of $\top$ (ohms).

$\rho'' = 1/y''$ . . . . . . resistance (A. C. impedance) of architrave of $\sqcap$ (ohms).

σ . . . . resistance (A. C. impedance) of impedance load (ohms).

$\theta, \theta_{ll}, \theta_1, \theta_2, \theta_3$ . . . . hyperbolic angle subtended by a single line, loop-line, and sections (hyps).

$y = 1/z$ . . . . surge-admittance (D. C. conductance) of a line (mhos).

$y' = 1/\rho'$ . . . . . . admittance (D. C. conductance) of a line-branch of a $\top$ (mhos).

$y'' = 1/\rho''$ . . . . . . admittance (D. C. conductance) of architrave of a $\sqcap$ (mhos).

u, u_A, u_P . . . . potential, at the sending-end, and at a point on the line (volts).

Z_r, Z_s . . . . impedance of a terminal receiver, of terminal sending apparatus (ohms).

z, z_{ll}, z_1, z_2, z_3 . . . . surge-impedance of a line, a loop-line, and sections (ohms).

z_0 . . . . apparent surge-impedance of a line to which an impedance load is prefixed (ohms).

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